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A property of $B_p(G)$. Applications to convolution operators

Antoine Derighetti

EPFL SB SMA-GE (Bâtiment MA), Station 8, CH-1015 Lausanne, Switzerland

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Abstract

A new property of $B_p(G)$, permits to obtain an approximation theorem for p-convolution operators and a non-commutative version of the Lohoué's monomorphism theorem concerning the norm closure of the set of all p-convolution operators with compact support.

Keywords: Convolution operators; Multipliers; Herz Figà-Talamanca algebra; Amenable groups

1. Introduction

Let G be a locally compact group and H a dense subgroup. Let $1 < p < \infty$ and $T \in CV_p(G)$. Suppose that G_d is amenable. We prove the existence of a net $(\mu_\alpha)_{\alpha \in I}$ of finitely supported measures such that:

(1) $\operatorname{supp} \mu_\alpha \subset H$,
(2) the norm of μ_α, considered as a convolution operator of $L^p(G)$, is not larger than the norm of T,
(3) the net of operators $(\mu_\alpha)_{\alpha \in I}$ converges strongly to T.

E-mail address: antoine.derighetti@epfl.ch.

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In 1971 N. Lohoué obtained this assertion assuming that the group G is abelian. Even for $G = \mathbb{R}$, $H = \mathbb{Q}$ and the convolution operator of $L^p(\mathbb{R})$ $T\varphi = f * \varphi$, with f a continuous function with compact support, this result is not trivial.

Let G be a locally compact amenable group, H a locally compact group and ω a continuous injective homomorphism of G into H. We show that for every bounded measure μ on G, the norm of μ, considered as a convolution operator of $L^p(G)$, is equal to the norm of $\omega(\mu)$, as a convolution operator of $L^p(H)$.

The above approximation theorem and the preceding monomorphism's theorem, are derived from the following property of the famous Banach algebra $B_p(G)$ introduced by C. Herz [7,8].

Theorem. *Let G be an arbitrary locally compact group, H a dense subgroup, $1 < p < \infty$ and u a complex valued continuous function on G with* $\mathrm{Res}_H u \in B_p(H_d)$. *Then $u \in B_p(G)$ and*

$$\|u\|_{B_p(G)} = \|\mathrm{Res}_H u\|_{B_p(H_d)}.$$

We briefly recall that $B_p(G)$ is the set of all pointwise multipliers of the Herz Figà-Talamanca algebra $A_p(G)$ if G is amenable. The Banach algebra $B_p(G)$ has been until recently intensively investigated by many authors (see for instance [13]). Even the case $p = 2$ is highly interesting. One of the reasons for this interest is that the Fourier algebra of many non-amenable groups G admits approximate units bounded in the norm of $B_2(G)$. In this paper, $B_p(G)$ appears as a tool for the investigation of the Banach algebra $CV_p(G)$ of all p-convolution operators of a locally compact group G. Our approach is new even for G abelian.

We say that a continuous operator T of $L^p(G)$ is a p-convolution operator if $T({}_a\varphi) = {}_aT\varphi$ for $a \in G$ and $\varphi \in L^p(G)$ where ${}_a\varphi(x) = \varphi(ax)$. In Section 2 we recall the definition of the Banach algebra $B_p(G)$.

The three main results of this paper are Theorem 7 (Section 2), Theorem 9 (Section 3) and Theorem 10 (Section 4).

2. A property of $B_p(G)$

Let X be a locally compact T_2 space, μ a positive Radon measure on X and $1 < p < \infty$. Suppose that $\mu(U) > 0$ for every non-empty relatively compact open subset U of X.

Let $k \in \mathbb{C}^{X\times X}$, $\mu \otimes \mu$-measurable and bounded. Suppose the existence of A, μ-integrable subset of X, with $\{(x, y) \mid k(x, y) \neq 0\} \subset A \times A$. For f, p-integrable the relation $T_k[f] = [g]$ with

$$g(x) = \int_X k(x, y) f(y)\, d\mu(y)$$

for every $x \in X$, defines a linear operator of $L^p_{\mathbb{C}}(X, \mu)$, we have

$$|||T_k|||_p \leqslant \|k\|_\infty \mu(A)$$

where $|||T_k|||_p$ is the bound of the operator T_k.

For $f \in \mathbb{C}^X$ we denote by $[f]$ the set $\{g \in \mathbb{C}^X \mid f(x) = g(x) \text{ almost everywhere}\}$ and for f, p-integrable ($f \in \mathcal{L}^p_{\mathbb{C}}(X, \mu)$) we put

$$N_p(f) = \left(\int_X \left| f(x) \right|^p d\mu(x) \right)^{1/p}.$$

$\mathcal{L}(L^p_{\mathbb{C}}(X,\mu))$ is the Banach space of all continuous operators of $L^p_{\mathbb{C}}(X,\mu)$.

For the definitions of $PM_p(G)$ (p-pseudomeasures) and $A_p(G)$ (Figà-Talamanca Herz algebra) for a locally compact group G, we refer to [5, pp. 232–233].

If $k \in C_{00}(X;\mathbb{C}) \otimes C_{00}(X;\mathbb{C})$ then T_k has finite range. For $k \in C_{00}(X \times X;\mathbb{C})$, T_k is in the norm $|||\,|||_p$ closure of $\{T_l \mid l \in C_{00}(X;\mathbb{C}) \otimes C_{00}(X;\mathbb{C})\}$ in $\mathcal{L}(L^p_{\mathbb{C}}(X,\mu))$. We precise that $C(X;\mathbb{C})$ is the set of all continuous maps of X into $\mathbb{C}$ and that $C_{00}(X;\mathbb{C})$ is the set $\{f \in C(X;\mathbb{C}) \mid \operatorname{supp} f \text{ is compact}\}$.

Lemma 1. *Let $k \in C_{00}(X \times X;\mathbb{C})$, $\varphi \in C(X \times X;\mathbb{C})$ and $\varepsilon > 0$. Then there are $U_1, \ldots, U_N$ open relatively compact subsets of X such that*:

(i) $\mathrm{pr}_1(\operatorname{supp} k) \cup \mathrm{pr}_2(\operatorname{supp} k) \subset U_1 \cup \cdots \cup U_N$,[1]
(ii) *for $1 \leqslant p, q \leqslant N$, for $x, x' \in U_p$ and for $y, y' \in U_q$ we have*

$$\left| \varphi(x,y)k(x,y) - \varphi(x',y')k(x,y) \right| < \varepsilon.$$

Proof. Let $K = \mathrm{pr}_1(\operatorname{supp} k) \cup \mathrm{pr}_2(\operatorname{supp} k)$, U an open relatively compact neighborhood of K and $r_1, \ldots, r_m, s_1, \ldots, s_m, t_1, \ldots, t_n, u_1, \ldots, u_n \in C_{00}(X;\mathbb{C})$ vanishing on $X \setminus K$ with

$$\left\| k - \sum_{i=1}^{m} r_i \otimes s_i \right\|_\infty < \frac{\varepsilon}{16C},$$

where

$$C = 1 + \sup\left\{ \left| \varphi(x,y) \right| \,\middle|\, (x,y) \in U \times U \right\}$$

and with

$$\left\| \varphi k - \sum_{i=1}^{n} t_j \otimes u_j \right\|_\infty < \frac{\varepsilon}{16}.$$

For every $x \in K$ there is $U_{(x)}$, open neighborhood of x with $U_{(x)} \subset U$, such that for every $x' \in U_{(x)}$, for every $1 \leqslant i \leqslant m$ and for every $1 \leqslant j \leqslant n$

$$\left| r_i(x) - r_i(x') \right| < \frac{\varepsilon}{16C(1 + \sum_{l=1}^{m} \|s_l\|_\infty)},$$

$$\left| s_i(x) - s_i(x') \right| < \frac{\varepsilon}{16C(1 + \sum_{l=1}^{m} \|r_l\|_\infty)},$$

$$\left| u_j(x) - u_j(x') \right| < \frac{\varepsilon}{16(1 + \sum_{l=1}^{n} \|t_l\|_\infty)},$$

[1] $\mathrm{pr}_1(x,y) = x$ and $\mathrm{pr}_2(x,y) = y$.

$$\left|t_j(x) - t_j(x')\right| < \frac{\varepsilon}{16(1 + \sum_{l=1}^{n} \|u_l\|_\infty)}.$$

There are $x_1, \ldots, x_N$ in K with $K \subset \bigcup_{p=1}^{N} U_{(x_p)}$. Then $U_1 = U_{(x_1)}, \ldots, U_N = U_{(x_N)}$ satisfy the required properties. □

Let $\mathcal{E}(X, \mu)$ be the set $\{(E_1, \ldots, E_n) \mid E_1, \ldots, E_n$ are disjoint Borel μ-integrable subsets of X with $\mu(E_i) > 0$ for $1 \leqslant i \leqslant n\}$.

We define on $\mathcal{E}(X, \mu)$ a non-filtering partial order $\delta \leqslant \delta'$ with $\delta = (E_1, \ldots, E_n)$ and $\delta' = (E'_1, \ldots, E'_{n'})$ by:

(1) for every $1 \leqslant j \leqslant n'$ there is $1 \leqslant i \leqslant n$ with $E'_j \subset E_i$,
(2) for every $1 \leqslant i \leqslant n$ there are P, μ-negligible and J a subset of $\{1, \ldots, n'\}$, with $E_i = P \sqcup \bigcup_{r \in J} E'_r$.

Lemma 2. *Let $k \in C_{00}(X \times X; \mathbb{C})$, $\varphi \in C(X \times X; \mathbb{C})$, $\varepsilon > 0$ and $\delta \in \mathcal{E}(X, \mu)$. Then there is $\delta_1 \in \mathcal{E}(X, \mu)$ such that*:

(i) $\delta \leqslant \delta_1$,
(ii) *for every $\delta' = (E'_1, \ldots, E'_{n'}) \in \mathcal{E}(X, \mu)$ with $\delta_1 \leqslant \delta'$ and for every $x'_1 \in E'_1, \ldots, x'_{n'} \in E'_{n'}$, we have*

$$\left|\varphi(x, y)k(x, y) - \varphi\left(x'_i, y'_j\right)k(x, y)\right| < \varepsilon$$

for $1 \leqslant i, j \leqslant n'$, $x \in E'_i$ and $y \in E'_j$.

Proof. Let $\delta = (E_1, \ldots, E_m)$, $K = \mathrm{pr}_1(\mathrm{supp}\, k) \cup \mathrm{pr}_2(\mathrm{supp}\, k)$, $I = \{i \mid 1 \leqslant i \leqslant m, \mu(E_i \cap K) > 0\}$ and $J = \{j \mid 1 \leqslant j \leqslant m,\ \mu(E_j \setminus K) > 0\}$. By Lemma 1 there are $U_1, \ldots, U_N$ open relatively compact subsets of X with $K \subset U_1 \cup \cdots \cup U_N$ and

$$\left|\varphi(x, y)k(x, y) - \varphi(x', y')k(x, y)\right| < \varepsilon$$

for $1 \leqslant p, q \leqslant N$, $x, x' \in U_p$ and $y, y' \in U_q$.

Let $\mathcal{F} = \{K \cap E_i \cap U_p \mid i \in I,\ 1 \leqslant p \leqslant N,\ \mu(K \cap E_i \cap U_p) > 0\} \cup \{E_j \setminus K \mid j \in J\}$ and $\delta_1 = (E_1^{(1)}, \ldots, E_n^{(1)}) \in \mathcal{E}(X, \mu)$ such that:

(1) for every $1 \leqslant i \leqslant n$ there is $F \in \mathcal{F}$ with $E_i^{(1)} \subset F$,
(2) for every $F \in \mathcal{F}$ there are N, μ-negligible and $\mathcal{G} \subset \{E_1^{(1)}, \ldots, E_n^{(1)}\}$ such that $F = N \cup \bigcup_{G \in \mathcal{G}} G$.

It is straightforward to verify that δ_1 satisfies the desired properties. □

Let $\delta = (E_1, \ldots, E_n)$ and $\delta' = (E'_1, \ldots, E'_{n'})$ elements of $\mathcal{E}(X, \mu)$. We say that $\delta \preccurlyeq \delta'$ if for every $1 \leqslant i \leqslant n$ there are P, μ-negligible and J a subset of $\{1, \ldots, n'\}$, with $E_i = P \sqcup \bigcup_{r \in J} E'_r$. The relation $\preccurlyeq$ is a filtering partial order on $\mathcal{E}(X, \mu)$. For $\delta = (E_1, \ldots, E_n) \in \mathcal{E}(X; \mu)$ and $f \in \mathcal{L}^p_{\mathbb{C}}(X, \mu)$ we put

$$P_\delta[f]=\sum_{j=1}^{n}\frac{\int_X f(x)1_{E_j}(x)\,d\mu(x)}{\mu(E_j)}[1_{E_j}].$$

Then $|||P_\delta|||_p=1$ and P_δ is a projection of $L^p_{\mathbb{C}}(X,\mu)$ onto

$$L^p_\delta(X,\mu)=\left\{\sum_{j=1}^{n}c_j[1_{E_j}]\;\middle|\;c_1,\dots,c_n\in\mathbb{C}\right\}.$$

Proposition 3. *For every compact operator T of $L^p(G)$, the net $(P_\delta T P_\delta)_{\delta\in\mathcal{E}(X,\mu),\preccurlyeq}$ converges strongly to T.*

Proof. Let $f\in\mathcal{L}^p_{\mathbb{C}}(X,\mu)$ and $\varepsilon>0$. There are $B_1,\dots,B_s$, μ-integrable Borel subsets of X with $\mu(B_i)>0$ for every $1\leqslant i\leqslant s$, $c_1,\dots,c_s\in\mathbb{C}$ such that

$$N_p(f-h)<\frac{\varepsilon}{4(1+|||T|||_p)}$$

where $h=\sum_{i=1}^{s}c_i1_{B_i}$. There is also T_0 operator of finite-dimensional range with $|||T-T_0|||_p<\frac{\varepsilon}{4(1+N_p(h))}$. For every $\varphi\in\mathcal{L}^p_{\mathbb{C}}(X,\mu)$ we have

$$T_0[\varphi]=\sum_{j=1}^{m}\left(\int_X\varphi(x)\varphi_j(x)\,d\mu(x)\right)[g_j],$$

where $g_1,\dots,g_m\in\mathcal{L}^p_{\mathbb{C}}(X,\mu)$ and $\varphi_1,\dots,\varphi_m\in\mathcal{L}^{p'}_{\mathbb{C}}(X,\mu)$.

For every $1\leqslant i\leqslant m$ there are:

(1) $n_i\in\mathbb{N}$,
(2) Borel sets $\{E_{ij}\mid 1\leqslant j\leqslant n_i\}$ with $\mu(E_{ij})\in(0,\infty)$,
(3) $\{d_{ij}\mid 1\leqslant j\leqslant n_i\}\subset\mathbb{C}$, such that

$$N_p(g_i-h_i)<\frac{\varepsilon}{8m(1+M)},$$

where

$$h_i=\sum_{j=1}^{n_i}d_{ij}1_{E_{ij}}$$

and

$$M=\max\left\{\left|\int_X h(x)\varphi_k(x)\,d\mu(x)\right|\;\middle|\;1\leqslant k\leqslant m\right\}.$$

There is $\delta=(E_1,\dots,E_n)\in\mathcal{E}(X;\mu)$ such that:

(1) for every $F \in \mathcal{F}$ $(= \{B_1, \ldots, B_s, E_{11}, \ldots, E_{1n_1}, \ldots, E_{mn_1}, \ldots, E_{mn_m}\})$ there are N, μ-negligible and $\mathcal{E}$ subset of $\{E_1, \ldots, E_n\}$ with $F = N \sqcup \bigcup_{E \in \mathcal{E}} E$,
(2) for every $1 \leqslant i \leqslant n$ there is $F \in \mathcal{F}$ with $E_i \subset F$.

Let $\delta' \in \mathcal{E}(X; \mu)$ with $\delta \preccurlyeq \delta'$. We have

$$\left\| T[f] - P_{\delta'} T P_{\delta'}[f] \right\|_p < \frac{3\varepsilon}{4} + \left\| T_0[h] - P_{\delta'} T_0 P_{\delta'}[h] \right\|_p,$$

but

$$\left\| T_0[h] - P_{\delta'} T_0 P_{\delta'}[h] \right\|_p \leqslant \sum_{j=1}^{m} \left| \int_X h(x)\varphi_j(x)\, d\mu(x) \right| \left\| [g_j] - P_{\delta'}[g_j] \right\|_p.$$

Taking in account that

$$\left\| [g_j] - P_{\delta'}[g_j] \right\|_p < \frac{\varepsilon}{4m(1+M)}$$

for every $1 \leqslant j \leqslant m$, we indeed obtain

$$\left\| T[f] - P_{\delta'} T P_{\delta'}[f] \right\|_p < \varepsilon. \qquad \square$$

For $x_1, \ldots, x_n \in X$ and $\varphi \in \mathbb{C}^{X \times X}$ we denote by $l(x_1, \ldots, x_n; \varphi)$ the matrix of $\mathbb{M}_n(\mathbb{C})$ $(\varphi(x_i, x_j))$. Let $\delta = (E_1, \ldots, E_n)$ be an element of $\mathcal{E}(X, \mu)$. The map

$$S_\delta : (a_1, \ldots, a_n) \mapsto \sum_{j=1}^{n} \frac{a_j}{\mu(E_j)^{1/p}} [1_{E_j}]$$

is a linear isometry of l_n^p onto $L_\delta^p(X, \mu)$. For $T \in \mathcal{L}(L_{\mathbb{C}}^p(X, \mu))$, we denote by $a_\delta(T)$ the matrix of the map $S_\delta^{-1} P_\delta T P_\delta S_\delta$ with respect to the canonical basis of l_n^p.

Lemma 4. *Let $k \in C_{00}(X \times X; \mathbb{C})$, $\varphi \in C(X \times X; \mathbb{C})$, $\varepsilon > 0$ and $\delta_0 \in \mathcal{E}(X, \mu)$. Then there is $\delta_1 \in \mathcal{E}(X, \mu)$ such that*:

(i) $\delta_0 \leqslant \delta_1$,
(ii) *for every $\delta' = (E'_1, \ldots, E'_{n'}) \in \mathcal{E}(X, \mu)$ with $\delta_1 \leqslant \delta'$ and for every $x'_1 \in E'_1, \ldots,$ for every $x'_{n'} \in E'_{n'}$, we have*

$$\left\| \left| a_{\delta'}(T_{\varphi k}) - l\left(x'_1, \ldots, x'_n; \varphi\right) * a_{\delta'}(T_k) \right| \right\|_p < \varepsilon,$$

where

$$l\left(x'_1, \ldots, x'_n; \varphi\right) * a_{\delta'}(T_k)$$

denotes the Hadamard product of the matrices $l(x'_1, \ldots, x'_n; \varphi)$ and $a_{\delta'}(T_k)$.

Proof. Let $\delta_0 = (E_1^{(0)}, \ldots, E_m^{(0)})$. According to Lemma 2 there is $\delta_1 \in \mathcal{E}(X, \mu)$ such that:

(i) $\delta_0 \leqslant \delta_1$,
(ii) for every $\delta' = (E_1', \ldots, E_{n'}') \in \mathcal{E}(X, \mu)$ with $\delta_1 \leqslant \delta'$, for every $x_1' \in E_1', \ldots, x_{n'}' \in E_{n'}'$, for every $1 \leqslant i, j \leqslant n'$, for every $x \in E_i'$ and for every $y \in E_j'$ we have

$$\left|k(x, y)\varphi(x, y) - k(x, y)\varphi\left(x_i', x_j'\right)\right| < \frac{\varepsilon}{2A},$$

where $A = \sum_{r=1}^{m} \mu(E_r^{(0)})$.

Let therefore $\delta' = (E_1', \ldots, E_{n'}')$ be an element of $\mathcal{E}(X, \mu)$ with $\delta_1 \leqslant \delta'$ and $x_1' \in E_1', \ldots, x_{n'}' \in E_{n'}'$. We put

$$d = (d_{ij}) = a_{\delta'}(T_{\varphi k}) - l\left(x_1', \ldots, x_n'; \varphi\right) * a_{\delta'}(T_k).$$

For $1 \leqslant i, j \leqslant n'$ we have

$$d_{ij} = \frac{\int_X \int_X 1_{E_i'}(x) 1_{E_j'}(y)(\varphi(x, y)k(x, y) - \varphi(x_i', x_j')k(x, y))\, d\mu(x)\, d\mu(y)}{\mu(E_i')^{1/p'} \mu(E_j')^{1/p}}$$

and consequently

$$|d_{ij}| \leqslant \frac{\varepsilon \mu(E_i')^{1/p} \mu(E_j')^{1/p'}}{2A}.$$

The inequality

$$|||d|||_p^p \leqslant \sum_{i=1}^{n'} \left(\sum_{j=1}^{n'} |d_{ij}|^{p'} \right)^{p/p'}$$

implies

$$|||d|||_p \leqslant \frac{\varepsilon}{2A} \sum_{j=1}^{n'} \mu\left(E_j'\right)$$

and finally $|||d|||_p < \varepsilon$. $\square$

Lemma 5. *Let $k \in C_{00}(X \times X; \mathbb{C})$, $\varphi \in C(X \times X; \mathbb{C})$ and $\varepsilon > 0$. Then there is $\delta_0 \in \mathcal{E}(X, \mu)$ such that for every $\delta \in \mathcal{E}(X, \mu)$ with $\delta_0 \leqslant \delta$ and for every $x_1 \in E_1, \ldots, x_n \in E_n$, where $\delta = (E_1, \ldots, E_n)$, we have*

$$|||T_{\varphi k}|||_p < \varepsilon + \left|\left|\left| l(x_1, \ldots, x_n; \varphi) * a_\delta(T_k) \right|\right|\right|_p.$$

Proof. By Proposition 3 there is $\delta_0 \in \mathcal{E}(X,\mu)$ such that for every $\delta \in \mathcal{E}(X,\mu)$ with $\delta_0 \preccurlyeq \delta$ we have

$$|||T_{\varphi k}|||_p - \frac{\varepsilon}{2} < |||P_\delta T_{\varphi k} P_\delta|||_p.$$

By Lemma 4 there is $\delta_1 \in \mathcal{E}(X,\mu)$ such that:

(i) $\delta_0 \leqslant \delta_1$,
(ii) for every $\delta = (E_1,\ldots,E_n) \in \mathcal{E}(X,\mu)$ with $\delta_1 \leqslant \delta$ and for every $x_1 \in E_1,\ldots,x_n \in E_n$ we have

$$\left|\left|\left|a_\delta(T_{\varphi k}) - l(x_1,\ldots,x_n;\varphi) * a_\delta(T_k)\right|\right|\right|_p < \frac{\varepsilon}{2}.$$

Let therefore $\delta = (E_1,\ldots,E_n) \in \mathcal{E}(X,\mu)$ with $\delta_1 \leqslant \delta$. Let also $x_1 \in E_1,\ldots,x_n \in E_n$. Then

$$|||T_{\varphi k}|||_p - \frac{\varepsilon}{2} < \left|\left|\left|a_\delta(T_{\varphi k})\right|\right|\right|_p < \frac{\varepsilon}{2} + \left|\left|\left|l(x_1,\ldots,x_n;\varphi) * a_\delta(T_k)\right|\right|\right|_p. \qquad \square$$

We recall now an important notion due to C. Herz [8, p. 150].

Definition 1. We say that $\varphi \in C(X \times X;\mathbb{C})$ belongs to $V_p(X \times X, \mu \otimes \mu)$, where $1 < p < \infty$, if there exists $C \in [0,\infty)$ such that

$$|||T_{\varphi k}|||_p \leqslant C|||T_k|||_p$$

for every $k \in C_{00}(X;\mathbb{C}) \otimes C_{00}(X;\mathbb{C})$. The smallest possible C is denoted $\|\varphi\|_{V_p(X\times X,\mu\otimes\mu)}$.

Lemma 6. *Let Y be a dense subset of X, $1 < p < \infty$ and $\varphi \in C(X \times X;\mathbb{C})$. Suppose that* $\mathrm{Res}_{Y\times Y}\,\varphi \in V_p(Y_d \times Y_d, m \otimes m)$*, where m is the counting measure of the topological space Y_d. Then $\varphi \in V_p(X \times X, \mu \otimes \mu)$ and*

$$\|\varphi\|_{V_p(X\times X,\mu\otimes\mu)} = \|\mathrm{Res}_{Y\times Y}\,\varphi\|_{V_p(Y_d\times Y_d,m\otimes m)}.$$

Proof. Let $k \in C_{00}(X;\mathbb{C}) \otimes C_{00}(X;\mathbb{C})$ and $\varepsilon > 0$. By Lemma 5 there is $\delta = (E_1,\ldots,E_n) \in \mathcal{E}(X,\mu)$ such that for every $x_1 \in E_1,\ldots,x_n \in E_n$

$$|||T_{\varphi k}|||_p < \frac{\varepsilon}{2} + \left|\left|\left|l(x_1,\ldots,x_n;\varphi) * a_\delta(T_k)\right|\right|\right|_p.$$

Let $0 < \eta < \frac{\varepsilon}{2(1+A^{1/p})}$ where

$$A = \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^{p'}\right)^{p/p'}$$

and $(a_{ij}) = a_\delta(T_k)$.

For every $1 \leqslant i, j \leqslant n$ there is W_{ij} open neighborhood of (x_i, x_j) in $X \times X$ with

$$\left|\varphi(x_i, x_j) - \varphi(x, y)\right| < \eta$$

for every $(x, y) \in W_{ij}$. Let W_i, open neighborhood of x_i, Z_j, open neighborhood of x_j, with $W_i \times Z_j \subset W_{ij}$ and finally $U_i = W_i \cap Z_i$. For every $1 \leqslant i \leqslant n$ there is $y_i \in Y \cap U_i$. Consequently for $1 \leqslant i, j \leqslant n$ we have

$$\left|\varphi(x_i, x_j) - \varphi(y_i, y_j)\right| < \eta.$$

We have therefore

$$\left|\!\left|\!\left|\big(l(x_1, \dots, x_n; \varphi) - l(y_1, \dots, y_n; \varphi)\big) * a_\delta(T_k)\right|\!\right|\!\right|_p < \frac{\varepsilon}{2},$$

this implies

$$\left|\!\left|\!\left| l(x_1, \dots, x_n; \varphi) * a_\delta(T_k)\right|\!\right|\!\right|_p < \frac{\varepsilon}{2} + \left|\!\left|\!\left| l(y_1, \dots, y_n; \varphi) * a_\delta(T_k)\right|\!\right|\!\right|_p$$

and consequently

$$|||T_{\varphi k}|||_p < \varepsilon + \|\mathrm{Res}_{Y \times Y}\, \varphi\|_{V_p(Y_d \times Y_d, m \otimes m)} |||T_k|||_p.$$

We obtain that $\varphi \in V_p(X \times X, \mu \otimes \mu)$ and

$$\|\varphi\|_{V_p(X \times X, \mu \otimes \mu)} \leqslant \|\mathrm{Res}_{Y \times Y}\, \varphi\|_{V_p(Y_d \times Y_d, m \otimes m)}.$$

Finally Lemma 2 of [8, p. 150] permits to conclude. □

Let us finally recall the definition of $B_p(G)$ proposed by C. Herz [8, p. 146].

Definition 2. Let G be an arbitrary locally compact group, $1 < p < \infty$ and $u \in C(G; \mathbb{C})$. We say that $u \in B_p(G)$ if $M_G u \in V_p(G \times G, m_G \otimes m_G)$ where m_G is a left-invariant measure of G and where $M_G f(x, y) = f(y^{-1}x)$ for $f \in \mathbb{C}^G$. For $u \in B_p(G)$ we put $\|u\|_{B_p(G)} = \|M_G u\|_{V_p(G \times G, m_G \otimes m_G)}$.

In [8], C. Herz proved, for G arbitrary locally compact group, that $B_p(G) = B_p(G_d) \cap C(G; \mathbb{C})$ and that $\|u\|_{B_p(G)} = \|u\|_{B_p(G_d)}$.

One of the main results of this paper is the following theorem.

Theorem 7. *Let G be an arbitrary locally compact group, H a dense subgroup,* $1 < p < \infty$ *and* $u \in C(G; \mathbb{C})$. *Suppose that* $\mathrm{Res}_H\, u \in B_p(H_d)$. *Then* $u \in B_p(G)$ *and*

$$\|u\|_{B_p(G)} = \|\mathrm{Res}_H\, u\|_{B_p(H_d)}.$$

Proof. We have $\mathrm{Res}_{H\times H}\circ M_G = M_H \circ \mathrm{Res}_H$. This implies $\mathrm{Res}_{H\times H} M_G u \in V_p(H_d \times H_d, m\otimes m)$. By Lemma 6 $M_G u \in V_p(G\times G, m_G \otimes m_G)$ and

$$\|M_G u\|_{V_p(G\times G, m_G\otimes m_G)} = \|\mathrm{Res}_{H\times H} M_G u\|_{V_p(H_d\times H_d, m\otimes m)}. \qquad \square$$

Remark. Even for G compact, the result is new.

3. Application to spectral synthesis

Let μ be a bounded complex measure on G ($\mu \in M^1_{\mathbb{C}}(G)$) then $\lambda^p_G(\mu)$ denotes the element of $CV_p(G)$ defined by

$$\lambda^p_G(\mu)[\varphi] = \left[\varphi * \Delta_G^{1/p'}\check{\mu}\right]$$

for $\varphi \in C_{00}(G;\mathbb{C})$, where Δ_G is the modular function of G. We precise that $\check{\mu}$ is the measure defined by $\check{\mu}(\varphi) = \mu(\check{\varphi})$ where $\check{\varphi}(x) = \varphi(x^{-1})$. The map λ^p_G is a faithful representation of the Banach algebra $M^1_{\mathbb{C}}(G)$ into $L^p(G)$. Therefore, via λ^p_G, every bounded measure on G can be considered as a p-convolution operator.

Using the commutation theorem and the Kaplansky's density theorem, we easily get the following result.

Proposition 8. *Let G be an arbitrary locally compact group and H a dense subgroup. For every $T \in CV_2(G)$ there is a net (μ_α) of complex measures on G such that*:

(i) $\operatorname{supp}\mu_\alpha$ *is finite,*
(ii) $\operatorname{supp}\mu_\alpha \subset H$,
(iii) $|||\lambda^2_G(\mu_\alpha)|||_2 \leqslant |||T|||_2$,
(iv) *the net $(\lambda^2_G(\mu_\alpha))$ converges strongly to T.*

We intend to generalize this proposition to $p \neq 2$. In [11, Théorème 1] N. Lohoué succeeded assuming G abelian.

Theorem 9. *Let G be a locally compact group and H a dense subgroup. Suppose that G_d is amenable. Let $1 < p < \infty$ and $T \in CV_p(G)$. Then there is a net (μ_α) of complex measures on G such that*:

(i) $\operatorname{supp}\mu_\alpha$ *is finite,*
(ii) $\operatorname{supp}\mu_\alpha \subset H$,
(iii) $|||\lambda^p_G(\mu_\alpha)|||_p \leqslant |||T|||_p$,
(iv) *the net $(\lambda^p_G(\mu_\alpha))$ converges strongly to T.*

Proof. It suffices to consider the case where $|||T|||_p = 1$. Let D be the closure in $CV_p(G)$, with respect to the strong operator topology, of the set of all convolution operators $\lambda^p_G(\mu)$, where μ is a complex measure with finite support contained in H with $|||\lambda^p_G(\mu)|||_{CV_p(G)} \leqslant 1$. Suppose that $T \notin D$. Then there is $u \in A_p(G)$ with $\langle u, T\rangle_{A_p(G),PM_p(G)} > 1$ and $|\langle u, S\rangle_{A_p(G),PM_p(G)}| \leqslant 1$ for every $S \in D$. For the definitions of $PM_p(G)$ and of the pairing $\langle\, , \rangle_{A_p(G),PM_p(G)}$, we again refer to [5, pp. 232, 233].

Theorem 2 of [8, p. 147] and Theorem 7 above imply

$$\|u\|_{A_p(G)} = \sup\bigl\{\bigl|\tilde{\mu}(u)\bigr| \bigm| \operatorname{supp}\mu \subset H,\ \operatorname{supp}\mu \text{ is finite},\ \bigl\|\bigl|\lambda^p_{H_d}(\mu)\bigr\|\bigr|_{CV_p(H_d)} \leqslant 1\bigr\}.$$

We precise that for every measure μ, $\tilde{\mu}(\varphi) = \overline{\mu(\tilde{\varphi})}$ where $\tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$ for $\varphi \in C_{00}(G;\mathbb{C})$.

For every measure μ with finite support in H we have

$$\bigl\|\bigl|\lambda^p_{H_d}(\mu)\bigr\|\bigr|_{CV_p(H_d)} = \bigl\|\bigl|\lambda^p_{G_d}(\mu)\bigr\|\bigr|_{CV_p(G_d)}$$

by [4, Théorème 1, p. 72]. But, by a recent result of Delmonico [1,3],

$$\bigl\|\bigl|\lambda^p_{G_d}(\mu)\bigr\|\bigr|_{CV_p(G_d)} = \bigl\|\bigl|\lambda^p_{G}(\mu)\bigr\|\bigr|_{CV_p(G)}.$$

Consequently $\|u\|_{A_p(G)} \leqslant 1$ and therefore $|\langle u, T\rangle_{A_p(G),PM_p(G)}| \leqslant 1$. This contradiction implies that $T \in D$. □

Remarks.

(1) See [1,3] for the case $H = G$. For $G = H = \mathbb{R}^n$ see [9].
(2) It is possible to have more informations on the net (μ_α) for certain classes of locally compact groups. See [10, Théorème II, p. 82] and [2].

4. The Lohoué's monomorphism theorem

Theorem 10. *Let G be a locally compact amenable group, H a locally compact group and ω a continuous injective homomorphism of G into H. Then for $1 < p < \infty$ and every bounded complex measure μ on G we have*

$$\bigl\|\bigl|\lambda^p_{G}(\mu)\bigr\|\bigr|_{CV_p(G)} = \bigl\|\bigl|\lambda^p_{H}\bigl(\omega(\mu)\bigr)\bigr\|\bigr|_{CV_p(H)}.$$

Proof. (1) Suppose at first that $\omega(G)$ is dense in H. Let ω^* be the map $u \mapsto u \circ \omega$. According to Theorem 7 ω^* is a linear isometry of $B_p(H)$ into $B_p(G)$. Let ω^{**} be the dual of ω^*. For $\mu \in M^1_{\mathbb{C}}(G)$ we put $F(u) = \tilde{\mu}(u)$ for $u \in B_p(G)$. We have $\|F\|_{B_p(G)^*} = |||\lambda^p_G(\mu)|||_p = \|\omega^{**}(F)\|_{B_p(H)^*} = |||\lambda^p_H(\omega(\mu))|||_p$.

(2) Suppose now that $\omega(G)$ is not dense in H. Let I be the closure of $\omega(G)$ in H. We denote by ω' the map ω considered as a continuous homomorphism of G into I. Let i be the inclusion of I into H. We have $i \circ \omega' = \omega$. For $\mu \in M^1_{\mathbb{C}}(G)$ we have $|||\lambda^p_H(\omega(\mu))|||_{CV_p(H)} = |||\lambda^p_I(\omega'(\mu))|||_{CV_p(I)}$ by [4]. The part (1) implies that $|||\lambda^p_I(\omega'(\mu))|||_{CV_p(I)} = |||\lambda^p_G(\mu)|||_{CV_p(G)}$. □

Remarks.

(1) C. Delmonico showed in [1], that the map ω extends canonically to the norm closure in $\mathcal{L}(L^p_{\mathbb{C}}(G))$ of the space of all p-convolution operators with compact support. Replacing in

the preceding proof, the operator $\lambda_G^p(\mu)$ by any $T \in CV_p(G)$ with compact support, we obtain the isometry for this larger class of operators.
(2) This theorem was first obtained by N. Lohoué in [12] for G and H both abelian.
(3) Theorem 10 does not require the amenability of G_d.

The following density property is also due to Lohoué [9] for G commutative.

Corollary 11. *Let G be a locally compact amenable group, H a locally compact group and ω a continuous injective homomorphism of G into H with $\omega(G)$ dense in H. Then for $1 < p < \infty$ the set $\{u \circ \omega \mid u \in A_p(H)$ with $\|u\|_{A_p(H)} \leqslant 1\}$ is dense in $\{v \mid \|v\|_{B_p(G)} \leqslant 1\}$ for the topology of uniform convergence on compact subsets of G.*

Proof. Lohoué's proof [12, pp. 146, 147], taking into account Theorem 10, adapts without change! □

Final remarks.

(1) This paper solves Problems 9.5, 9.6, 9.10 and 9.12 of [6].
(2) For G abelian, our approach avoids the use of the structure theory of locally compact abelian groups in Theorems 7, 9 and 10.
(3) Even for $p = 2$, Theorems 7 and 10 are new.

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